Higher Algebraic *K*-theory and Representations of Algebraic Groups[1]

By

Aderemi Kuku, Ph.D, FAMS(USA), FTWAS, FAAS, FNMS, OON, NNOM
Immediate Past President, African Academy of Sciences, and Distinguished Professor of Mathematics, National Mathematical Centre, Abuja, Nigeria

Abstract

The talk starts with an attempt to popularize some original aims of K-theory, namely to classify various mathematical objects and structures and more generally the classifications of 'equivalence' classes of objects of various 'nice' categories $\mathscr{C}$, e.g. the category $\mathscr{P}(F)$ of finite dimensional vector spaces over fields F (or more generally the category $\mathscr{P}(R)$) of finitely generated projective modules over rings R with identity,(e.g. Dedekind domains such as rings of integers in number fields and p-adic fields; Banach and C*- algebras, Orders and group rings.): the category $\mathscr{VB}(X)$ of complex vector bundles over a compact space X: the category of algebraic vector bundles on a scheme X (or equivalently, the category of locally free sheaves of O_X -modules), etc.. Moreover, K-theory maps into or coincides with some other interesting theories, e.g. Galois, Etale, Motivic, DeRaum, Hochschild, Cyclic, (co)-homology theories; Homology of classical groups or arithmetic groups etc

We next note that representations of various groups G could be considered as actions of G on such 'nice' categories as are mentioned above giving rise to the category $\mathscr{C}_G$ of G-representations in C otherwise known as a G-equivariant category, on which one can do K-theory.

The initial motivation for this approach to representation theory is due to the fact that when $\mathscr{C} = \mathscr{P}(\mathbb{C})$, ($\mathbb{C}$ the field of complex numbers), G a finite or compact Lie group, the Grothendieck group $K_0\left(\mathscr{P}(\mathbb{C}_G\right)$ coincides with the Abelian group of generalized characters of G. As such, the study of K-theory of such equivariant categories belongs to the theory of group representations and are aptly described as Equivariant K-theory.

The categories $\mathscr{C}$ identified above are examples of "exact categories" which we next define in the talk as well as introduce equivariant exact categories with examples relevant for the action of algebraic groups.

So, for an algebraic group G over a field F, we present constructions and computations of equivariant higher K-groups as well as profinite (continuous) equivariant higher K-groups for some G-schemes when F is a number field or p-adic field.

For example, let ${}_{\gamma}\mathscr{F}$ be a twisted flag variety, B a finite dimensional separable F-algebra, and ℓ an odd rational prime. When F is a number field, we prove that for all

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$n \geq 1$, $K_{2n}^{\text{pr}}\left(\left({}_{\gamma}\mathscr{F}, B\right), \hat{\mathbb{Z}}_{\ell}\right)$ is an ℓ-complete Abelian group and $\operatorname{div} K_{2n}^{\text{pr}}\left(\left({}_{\gamma}\mathscr{F}, B\right), \hat{\mathbb{Z}}_{\ell}\right) = 0$. If F is a p-adic field, we prove that for $n \geq 1$, $K_{n}^{\text{pr}}\left(\left({}_{\gamma}\mathscr{F}, B\right), \hat{\mathbb{Z}}_{\ell}\right) \approx K_{n}\left(\left({}_{\gamma}\mathscr{F}, B\right), \hat{\mathbb{Z}}_{\ell}\right)$ are ℓ-complete profinite Abelian groups and $\operatorname{div} K_{n}^{\text{pr}}\left(\left({}_{\gamma}\mathscr{F}, B\right), \hat{\mathbb{Z}}_{\ell}\right) = 0$. As preliminary results, we prove that when F is a number field, then for the ordinary higher K-theories, we have $K_{2n+1}\left({}_{\gamma}\mathscr{F}, B\right)$ are finitely generated Abelian group and $K_{2n}\left({}_{\gamma}\mathscr{F}, B\right)$ are torsion while if F is a p-adic field, then for all $n \geq 2$, $K_{n}\left({}_{\gamma}\mathscr{F}, B\right)_{\ell}$ are finite groups.